Path integral in QFT:

ln Z_0 for Bosons & Fermions

Bosons

$$Z_0 = \int D\varphi \; e^{-\int d\tau \, \mathcal{L}_0^E}$$

$$\mathcal{L}_0^E = \frac{1}{2}\varphi\left(-\partial_\tau^2 + \omega^2\right)\varphi$$

$$\hat{D} \leftrightarrow -\nabla^2 + M^2$$

$$\sim \frac{1}{\sqrt{\det \hat{D}}}$$

$\sqrt{}^{-1} \longrightarrow$ real, bosonic φ

$\det \hat{D}$ = product of eigenvalues

$$\prod_{n=-\infty}^{+\infty} \omega_n^2 + \omega^2$$

PBC $\quad \omega_n \longrightarrow \frac{2n\pi}{\beta}$.

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$$\ln Z_0 = -\frac{1}{2} \ln \det \hat{D}$$

$$= -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \ln\left[\omega_n^2 + \omega^2\right]$$

trick #1

$$\int_{ref.}^{\beta\omega} dx' \coth\frac{x'}{2}$$

$$= \beta\omega + 2\ln(1-e^{-\beta\omega})$$

$$\ln Z_0 = -\frac{1}{2}\beta\omega - \ln\left[1-e^{-\beta\omega}\right] \quad //$$

trick #1

$$\sin(x) = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$\sinh(x) = x \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2\pi^2}\right)$$

$$\frac{\partial}{\partial x}\ln\sinh(x) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{n^2\pi^2+x^2}$$

$$\frac{\coth x}{x} = \sum_{n=-\infty}^{+\infty}\frac{1}{n^2\pi^2+x^2} \quad //$$

$$\frac{\partial}{\partial x^2}\sum_{n=-\infty}^{+\infty}\ln(n^2\pi^2+x^2)$$

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thus ..

$$\sum \ln (n^2\pi^2 + x^2) = \int^{x} dx' \; 2 \coth x'$$

$$= \int^{2x} dx' \; \coth \frac{x'}{2}$$

$$= (2x) + 2 \ln (1 - e^{-2x})$$

//

$$x \to \frac{1}{2}\beta\omega$$

$$\sum \ln \left[n^2\pi^2 + \left(\frac{\beta\omega}{2}\right)^2 \right] \longleftrightarrow \sum \ln \beta^2 \left[\frac{n^2\pi^2}{\beta^2} + \left(\frac{\omega}{2}\right)^2 \right]$$

$$\updownarrow$$

$$\beta\omega + 2 \ln (1 - e^{-\beta\omega}) + C$$

$$Z_0 = \frac{1}{\sqrt{\det(-\partial_\tau^2 + \omega^2)}} \Rightarrow \ln Z_0 = -\frac{1}{2} \mathrm{tr} \ln (-\partial_\tau^2 + \omega^2)$$

$$\updownarrow$$

$$-\frac{1}{2}\beta\omega - \ln \left[1 - e^{-\beta\omega} \right]$$

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$$\ln Z_0 \longrightarrow -\frac{1}{2}\,\mathrm{tr}\ln\left[-\partial_\tau^2-\nabla^2+m^2\right]$$

$$= -\frac{1}{2}\underline{\sum_n}\;\underline{\underline{V\int\frac{d^3k}{(2\pi)^3}}}\;\ln\Big(\omega_n^2+\underbrace{\vec{k}^2+m^2}_{\mathcal{E}_k^2}\Big)$$

$$(\beta V)\frac{1}{\beta}\sum\!\!\!\!\int\frac{d^3k}{(2\pi)^3}\left\{-\frac{1}{2}\ln\left[\omega_n^2+\mathcal{E}_k^2\right]\right\}$$

$$= (\beta V)\int\frac{d^3k}{(2\pi)^3}\left\{-\frac{1}{2}\mathcal{E}_k - T\ln\left[1-e^{-\beta\mathcal{E}_k}\right]\right\}$$

$$\mathcal{E}_k = \sqrt{\vec{k}^2+m^2}$$

//

$$\ln Z_0 \leftrightarrow -\beta V f$$

$\mathcal{E}_{vac}$

$$f_B = \underline{\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\mathcal{E}_k}$$

$$+\underline{-\int\frac{d^3k}{(2\pi)^3}\left\{-T\ln\left(1-e^{-\beta\mathcal{E}_k}\right)\right\}}$$

P_B.

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Fermions

$$Z_0 = \int D\psi D\bar{\psi}\, e^{\int \bar{\psi}(i\partial\!\!\!/ - m)\psi}$$

$$= \det[i\partial\!\!\!/ - m].$$

how <u>det</u> comes out ?

Grassmanian variables

$$\int d\eta = 0 \qquad \eta^2 = 0$$

$$\int d\eta\, \eta = 1 \qquad f(\eta) \to \underbrace{a + b\eta}_{\text{only}}$$

$$\int d\eta\, d\bar{\eta}\; e^{\bar{\eta} a \eta} = a.$$

$$\int d\eta_1 d\bar{\eta}_1 d\eta_2 d\bar{\eta}_2\; \underline{e^{\bar{\eta}_i A_{ij} \eta_j}}$$

$$\underbrace{}_{\text{only}} \; \bar{\eta}A\eta\; \bar{\eta}A\eta \to \text{can be non-zero}$$

$$\bar{\eta}_1 A_{11} \eta_1\, \bar{\eta}_2 A_{22} \eta_2 + \bar{\eta}_1 A_{12} \eta_2\, \bar{\eta}_2 A_{21} \eta_1$$

$$\to A_{11} A_{22} - A_{12} A_{21} = \det A$$

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$$\ln Z_0 = \ln \det (i\not\partial - m)$$

$$\omega_n = \frac{(2n+1)\pi}{\beta}$$

$$\rightarrow \mathrm{tr} \ln (i\not\partial - m)$$

$$= \beta V \frac{1}{\beta} \sum \int \frac{d^3k}{(2\pi)^3} \mathrm{tr} \ln \left[i\omega_n \gamma^0 - \vec{k} \cdot \vec{\gamma} - m \right]$$

Dirac trace

$$\mathrm{tr} \ln \left[\not{K} - m \right]$$

$$= \mathrm{tr} \ln \left[\gamma_5 (\not{K} - m) \gamma_5 \right]$$

$$= \mathrm{tr} \ln (-\not{K} - m)$$

$$\mathrm{tr} \ln (\not{K} - m)$$

$$\rightarrow \frac{1}{2} \mathrm{tr} \ln \left[(\not{K} - m)(-\not{K} - m) \right]$$

$$= \frac{1}{2} \mathrm{tr} \ln (K^2 - m^2) + C$$

$$\frac{1}{2} \mathrm{tr} I \ln (k^2 - m^2)$$

Dirac trace

$\rightarrow 4$: $p\,\bar{p}$, spin

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$$\ln Z_0 \to \beta V \quad \frac{1}{\beta}\sum\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \ln\left[\omega_n^2 + \varepsilon_i^2\right]$$

$$\left(4 \times \frac{1}{2}\right)$$

at finite μ $\qquad \mu N \to \mu \bar{\psi}\gamma^0\psi$

$$i\partial\!\!\!/ - m \to i\partial\!\!\!/ - m + \mu\gamma^0$$

$$\to \underline{-\partial_\tau\gamma^0} - i\nabla\cdot\vec{\gamma} - m \underline{+\mu\gamma^0}$$

$$i\omega_n\gamma^0$$

$$i\omega_n \to i\omega_n + \mu.$$

$$\ln Z_0 = \beta V \quad \frac{1}{\beta}\sum\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \; 4 \times \frac{1}{2} \times$$

$$\ln\left[(\omega_n - i\mu)^2 + \varepsilon_i^2\right]$$

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$$\ln\left[(\omega_n - i\mu)^2 + \varepsilon_p^2\right]$$

$$= \int_{\text{Ref}}^{\varepsilon} d\varepsilon' \frac{2\varepsilon'}{(\omega_n - i\mu)^2 + \varepsilon'^2}$$

Given

$$\frac{1}{\beta}\sum_{\omega_n} \frac{1}{(\omega_n - i\mu)^2 + \varepsilon'^2}$$

$$\rightarrow \frac{1}{2\varepsilon'}\left[1 - n(\varepsilon'+\mu) - n(\varepsilon'-\mu)\right]$$

$$n(\varepsilon') = \frac{1}{e^{\beta\varepsilon'} + 1}$$

$$\frac{1}{\beta}\sum_{\omega_n} \ln\left[(\omega_n - i\mu)^2 + \varepsilon_p^2\right]$$

$$= \int_{\text{Ref}}^{\varepsilon_p} d\varepsilon' \left[1 - n(\varepsilon'+\mu) - n(\varepsilon'-\mu)\right]$$

$$= \varepsilon_p + T\ln\left[1 + e^{-\beta(\varepsilon_p - \mu)}\right]$$
$$+ T\ln\left[1 + e^{-\beta(\varepsilon_p + \mu)}\right]$$

spin
$p, \bar{p}$

$\frac{1}{2} \times \mathrm{tr}_D \mathbb{I}$

$$\ln Z_0 = \beta V \times 2 \times$$

$$\int \frac{d^3k}{(2\pi)^3} \Big\{ \varepsilon_{\vec{k}} + T \ln\left[1 + e^{-\beta(\varepsilon_k - \mu)}\right]$$

$$+ T \ln\left[1 + e^{-\beta(\varepsilon_{\vec{k}} + \mu)}\right] \Big\}$$

vac

the − sign
for fermions

$$f_F = 2 \times$$

$$\int \frac{d^3k}{(2\pi)^3} \Big\{ -\varepsilon_{\vec{k}} +$$

$$- T \ln\left[1 + e^{-\beta(\varepsilon_k - \mu)}\right]$$

$$- T \ln\left[1 + e^{-\beta(\varepsilon_k + \mu)}\right] \Big\}$$

Pressure is "+" still

Contour Integral

$$\frac{1}{\beta}\sum \frac{-1}{(i\omega_n+\mu)^2-\varepsilon'^2} = \frac{1}{2\varepsilon'}\left(1-n-\bar{n}\right)$$

pole generator:

$$n(z) = \frac{1}{e^{\beta z}+1} \quad \text{or} \quad \frac{1}{2}\tanh\left(\frac{1}{2}\beta z\right)$$

↳ residue: $\frac{1}{\beta}$

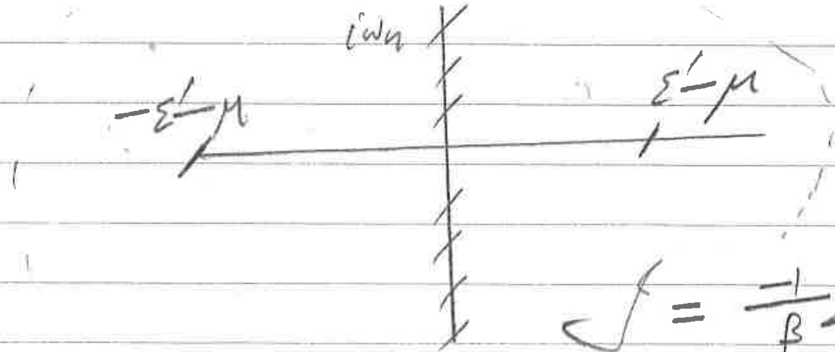
poles

$$z \longrightarrow i\omega_n = i\,\frac{(2n+1)\pi}{\beta}.$$

residue

$$\left(\frac{d}{dz}n^{-1}\right)^{-1}_{z\to i\omega_n} \longrightarrow \frac{-1}{\beta}$$

$$\mathcal{J} = 0 = \oint_{L} \frac{dz}{2\pi i}\,\frac{1}{(z+\mu)^2-\varepsilon'^2}\,n(z)$$



$$\mathcal{J} = \frac{-1}{\beta}\sum \frac{1}{(i\omega_n+\mu)^2-\varepsilon'^2} + \frac{1}{2\varepsilon'}\,n(\varepsilon'-\mu) + \frac{-1}{2\varepsilon'}\,n(-\varepsilon'-\mu)$$

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where

$$n(-\varepsilon-\mu) = \frac{1}{e^{-\beta(\varepsilon+\mu)}+1}$$

$$= 1 - \bar{n}(\varepsilon)$$

$$\frac{1}{\beta}\sum \frac{-1}{(i\omega_n+\mu)^2-\varepsilon^2} = \frac{1}{2\varepsilon}\left[1-n(\varepsilon)-\bar{n}(\varepsilon)\right]$$

//

$$\frac{1}{\beta}\sum \frac{1}{(\omega_n-i\mu)^2+\varepsilon^2}$$

see
PhD thesis.

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Of course, Euler's formula will work!

$$\cos(x) = \prod_{n=0}^{\infty} \left[1 - \frac{x^2}{(n+\frac{1}{2})^2 \pi^2} \right]$$

$$\cosh(x) = \prod_{n=0}^{\infty} \left[1 + \frac{x^2}{(n+\frac{1}{2})^2 \pi^2} \right]$$

$$\frac{\partial}{\partial x} \ln \cosh(x) = \sum_{n=0}^{\infty} \frac{2x}{(n+\frac{1}{2})^2 \pi^2 + x^2}$$

$\tanh(x)$

factor of 2 is perfect

$$\frac{\tanh(x)}{x} = \sum_{n=-\infty}^{+\infty} \frac{1}{(n+\frac{1}{2})^2 \pi^2 + x^2}$$

$$= \frac{\partial}{\partial x^2} \sum \ln \left[(n+\tfrac{1}{2})^2 \pi^2 + x^2 \right]$$

$$\sum_{n=-\infty}^{+\infty} \ln \left[(n+\tfrac{1}{2})^2 \pi^2 + x^2 \right] = \int_{\text{Ref}}^{x} dx' \, 2 \tanh x'$$

$$= \int^{2x} dx' \, \tanh \tfrac{1}{2} x'$$

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$$\sum_{n=-\infty}^{+\infty} \ln\left[(n+\tfrac{1}{2})^2\pi^2 + x^2\right] = \int^{2x} dx' \tanh \tfrac{1}{2}x'$$

$$= 2 \ln \cosh x$$

$$= 2x + 2 \ln\left[1 + e^{-2x}\right] + C$$

$$x = \frac{\beta \varepsilon_k}{2}$$

$$\frac{1}{\beta} \sum \ln\left[\frac{(2n+1)^2\pi^2}{\beta^2} + \varepsilon_k^2\right]$$

$$= \varepsilon_k + 2T \ln\left[1 + e^{-\beta\varepsilon_k}\right]$$

//

<u>it works too</u>

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Summary

$$\ln Z_B^{(0)} = \beta V \int \frac{d^3k}{(2\pi)^3} \Big\{ -\tfrac{1}{2}\varepsilon_{\vec{k}} + \\ - T \ln\left(1 - e^{-\beta \varepsilon_{\vec{k}}}\right) \Big\}$$

$p, \bar{p}$ & spin } Dirac trace

$$\ln Z_F^{(0)} = \beta V \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \times 4 \times \\ \Big\{ \varepsilon_{\vec{k}} + T \ln\left[1 + e^{-\beta(\varepsilon_k - \mu)}\right] \\ + T \ln\left[1 + e^{-\beta(\varepsilon_k + \mu)}\right] \Big\}$$

focus on the finite T piece : //

$$p \to \frac{(\ln Z)_{FT}}{\beta V}$$

Good checks $M \to 0$

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$$P_B \to \frac{\pi^2}{90} T^4$$

$$P_F \to \frac{7}{180}\pi^2 T^4 + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2}$$

$p, \bar{p}$
spin.

$\Downarrow$

per "component":

$$\underbrace{\frac{7}{8}}\left(\frac{\pi^2}{90} T^4\right)$$

famous factor of $7/8$
between F & B

Another check: $T = 0 \quad M \neq 0$

$$P_F \to \frac{1}{24\pi^2}\left\{ \mu p_F (2\mu^2 - 5M^2) + 3M^4 \ln\left(\frac{\mu + p_F}{M}\right) \right\}$$

$$\mu = \sqrt{p_F^2 + M^2}$$

$$n_F \to 2 \frac{1}{6\pi^2} p_F^3$$

w spin degen

~~Anti-particles contribute!~~ X

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$$P_F\,(T=0 \quad M\to 0) = \frac{\mu^4}{12\pi^2}$$ ✓

but the finite T : $p, \bar{p}$; spin

$$P_F\,(T\to 0 \quad M=0) = \frac{\mu^4}{12\pi^2}$$ just luck ?

What happens ?

at $T=0$; Physics : anti-part X contribute
spin degen only

$$P_F \to 2\times\frac{\mu^4}{24\pi^2} \quad \begin{matrix}(M=0)\\ T=0\end{matrix}$$

$$\ln Z_B^{(0)} \longrightarrow -\frac{1}{2}\,\mathrm{tr}\,\ln\left[\partial^2+m^2\right] \qquad + E_{vac}$$

$$\ln Z_F^{(0)} \longrightarrow \mathrm{tr}_D\,\ln\,(i\not\partial-m) \qquad - E_{vac}.$$

$$\to \frac{1}{2}\,\underline{\mathrm{tr}_D}\,\ln\left[\partial^2+m^2\right]$$

↳ fac. of 4 //

$$\left.\begin{matrix}\frac{1}{2} \to \mathrm{sqrt} \\ \mp \to B, F\end{matrix}\right\}$$ most beautiful formula

most beautiful eq. in QFT

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Alt. form for Pressure

$$P_B = T\int\frac{d^3k}{(2\pi)^3}\, -\ln\left[1-e^{-\beta\varepsilon_k}\right]$$

$$= \frac{-1}{2\pi^2}\,T\int_0^\infty dk\; k^2\;\ln\left[1-e^{-\beta\varepsilon_k}\right]$$

$$= \frac{-1}{2\pi^2}\,T\int_0^\infty dk\;\left[\frac{d}{dk}\left(\frac{k^3}{3}\right)\right]\;\ln\left[1-e^{-\beta\varepsilon_k}\right]$$

$$\int_0^\infty dk\;\frac{k^3}{3}\;-\frac{d}{dk}\;\frac{e^{-\beta\varepsilon_k}}{1-e^{-\beta\varepsilon_k}}\;\beta\;\frac{d\varepsilon_k}{dk}$$

$$\longrightarrow \frac{1}{2\pi^2}\int_0^\infty dk\;\frac{1}{3}k^3\frac{d\varepsilon_k}{dk}\;\frac{1}{e^{\beta\varepsilon_k}-1}$$

$$\frac{1}{3}k^3\,\frac{k}{\varepsilon_k}$$

$$= \frac{1}{2\pi^2}\int dk\;\frac{1}{3}k^3\frac{k}{\varepsilon_k}\;n_B(\varepsilon_k)$$

better convergence

$$P_F \longrightarrow \frac{1}{2\pi^2}\int dk\;\frac{1}{3}k^3\frac{k}{\varepsilon_k}\;n_F(\varepsilon_k)$$

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$p, \bar{p}$; $\frac{1}{2}$

$$P_\xi = \frac{1}{2}(4) \times \frac{1}{2\pi^2} \int dk\, k^2\, T \left[\ln\left[1 + e^{-\beta(\epsilon-\mu)}\right] + \ln\left[1 + e^{-\beta(\epsilon+\mu)}\right] \right]$$

↓ more useful form

~~Spin~~

$$2 \times \frac{1}{2\pi^2} \int dk\, \frac{1}{3} k^2 \frac{k}{E_k} \left[n_\xi(\epsilon_k) + \bar{n}_\xi(\epsilon_k) \right]$$

$$n_\xi(\epsilon_k) = \frac{1}{e^{\beta(\epsilon_k-\mu)}+1} \qquad \bar{n}_\xi(\epsilon_k) = \frac{1}{e^{\beta(\epsilon_k+\mu)}+1}$$

$M \to 0$

$$= \frac{1}{12 T_1^2} P_\xi^4$$

$T \to 0$, $\beta \to \infty$

$\theta(\epsilon_k - \mu)$ effective!

$\cancel{\theta(\epsilon_k+\mu)}$ No chance for Anti-particle

$= \{$ has spin degen 4 only particle can contribute $\}$

$$\frac{\partial}{\partial \mu} P_\xi \to n_\xi = 2 \times \left(\frac{P_\xi^3}{6\pi^2} \right)$$

is a familiar result ; in fact

⟶ general result for any M

First contact with
Fermi energy / surface :

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$$P_F^{T=0} = 2 \times \frac{1}{6\pi^2} \int dk \; k^3 \frac{d\varepsilon_k}{dk} \; \theta[\mu - \varepsilon_k]$$

// a natural UV cutoff for momentum //

$$n_F^{T=0} = \frac{\partial}{\partial \mu} P_F^{T=0}$$

$$= 2 \times \frac{1}{6\pi^2} \int dk \; k^3 \frac{d\varepsilon}{dk} \; \underline{\delta(\mu - \varepsilon_k)}$$

$$\frac{1}{\left|\frac{d\varepsilon}{dk}\right|} \delta(k - k_F)$$

Famous result:

$$\rightarrow 2 \times \frac{1}{6\pi^2} k_F^3$$

this derives

$$\mu = \sqrt{k_F^2 + M^2}$$

$$\Updownarrow$$

$$P_F^{T=0} = \int_M^{\mu} d\mu' \; n_F^{T=0}(\mu')$$

$$= 2 \times \frac{1}{48\pi^2} \Big\{ \mu k_F (2\mu^2 - 5M^2) + 3M^4 \ln \frac{\mu + k_F}{M} \Big\}$$

//

Note that : $P(T\to 0, \mu, M) \leftrightarrow P^{T=0}_{\mu,M}$ is very natural

$\Rightarrow$ check it Numerically!

$T \sim 10$ MeV is close
$\sim T \to 0$ situation

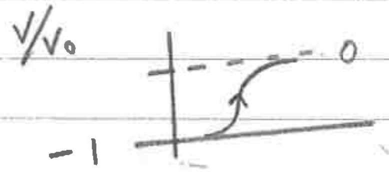
$$\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \longrightarrow \theta(\varepsilon-\mu)$$

is very effective

$\Downarrow$

Wood-Saxon potential $\Leftarrow$ Numerically useful tool!

$$-\frac{V_0}{1+e^{\frac{r-R}{a}}}$$



$$\frac{1}{e^{\beta(\varepsilon+\mu)}+1} \longrightarrow 0$$

essentially