HW1 due 27/10/2023

(+2 points if you solve the problems with Julia)

Q.1

Root finding.

- There are many methods to find the roots of a function: e.g.
  - just plot it.
  - bisection.
  - iterative method: Newton's, secant, etc.

  Pick one and solve for the **positive** root of $f(x) = x^2 - x - 1$. Of course the quadratic formula tells us the solution is

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

  (take the + root.) This is the Golden Mean.

- Show that the Golden Mean x satisfies

$$x = 1 + 1/x.$$

This provides yet another numerical scheme to solve for it, via a recursive formula:

$$\begin{aligned} x_1 &\to 1 \\ x_2 &\to 1 + 1/x_1 \\ x_3 &\to 1 + 1/x_2 \\ &\ldots \end{aligned}$$

Write a recursive function to implement this scheme.

Q.2

Finding inverses.

- Suppose we want to find the multiplicative inverse of a real number a. The answer is, of course, $1/a$. A fancy way to put it is that we want to solve for x such that

$$f(x) = 1/x - a = 0.$$

This becomes a root finding problem. Shows that the Newton's method gives

$$x_{n+1} = x_n\,(2 - a\,x_n).$$

Plug in numbers to verify that it is true.

- Show that the formula works also for finding an inverse of a matrix A.

$$X_{n+1} = X_n\,(2\,I - A\,X_n).$$

Demonstrate this with a 2x2 matrix. (Not all will work, good luck!)

Q.3

Complex **i** as a 2x2 matrix.

If we represent the complex number "i" as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

In Julia, you can write:

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imat = [ 0 -1; 1 0 ]
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Show how

$$\begin{aligned} i^2 &= -1 \\ exp(i\pi) &= -1 \\ exp(i\pi/2) &= i \end{aligned}$$

are realized.

Q.4

- Write a program to compute the Gaussian integral:

$$\int_{-\infty}^{\infty} dx\, e^{-A\,x^2}.$$

Check with the exact result

$$\sqrt{\frac{\pi}{A}}.$$

- If we define

$$\langle \hat{O}p \rangle = \frac{\int_{-\infty}^{\infty} dx\, e^{-(A/2)\,x^2}\, Op(x)\, e^{-(A/2)\,x^2}}{\int_{-\infty}^{\infty} dx\, e^{-A\,x^2}}.$$

Compute (numerically and analytically)

$$\langle x \rangle$$
$$\langle x^2 \rangle$$
$$\langle p \rangle = \langle -i\frac{d}{dx} \rangle$$
$$\langle p^2 \rangle = \langle -\frac{d^2}{dx^2} \rangle.$$

Show that

$$(\langle p^2 \rangle - \langle p \rangle^2) \times (\langle x^2 \rangle - \langle x \rangle^2) = \frac{1}{4}.$$

Does it smell like quantum mechanics?

Q.5

Consider the logistic map:

$$x_{n+1} = r x_n \,(1 - x_n).$$

- Produce the bifurcation diagram: for a given r within $(0, 4)$, iterate (starting from any initial value within $(0, 1)$), collect the end point(s) and plot them as a function of r.
- Derive an analytic result for the stable end points within $r < 3$. Plot it on top of the bifurcation diagram.
- For $r = 3.6$, collect all the points $\{x_j\}$ from iterations and plot them in a normalized histogram. This is called the invariant density.
- Plot the invariant density at $r \to 4$. Compare with the analytic result:

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$