HW2 due 17/11/2023

(+2 points if you solve the problems with Julia + on time)

Q.1

Pauli matrices.

The Pauli matrices are given by:

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sigma1 = [ 0 1; 1 0 ]
sigma2 = [ 0 -im; im 0 ]
sigma3 = [ 1 0; 0 -1 ]
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a) Prove the relations

$$(\vec{\sigma} \cdot a)(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}).$$

and

$$D(\phi, \hat{n}) = e^{-i\frac{1}{2}\vec{\sigma}\cdot\hat{n}\,\phi} = \cos\frac{\phi}{2}\, I_2 - i\sin\frac{\phi}{2}\vec{\sigma} \cdot \hat{n}.$$

($\hat{n}$ is a unit 3-vector.)

Verify these with a program.

b) If we write

$$D(\phi, \hat{n}) = \begin{bmatrix} z1 & z2 \\ z3 & z4 \end{bmatrix},$$

what are the restrictions on the complex numbers z_1, z_2, z_3, z_4? How many degrees of freedom do we have?

c) Write a program to generate random SU(2) matrices, i.e. matrices like $D(\phi, \hat{n})$.

d) Write a program to compute the Levi-Civita symbol (3D) via

$$\epsilon(i, j, k) = \frac{\mathrm{tr}\{[\sigma_i, \sigma_j], \sigma_k\}}{8i}.$$

(This is definitely one of the stupidest method.) Can you think of one other method to do this?

Q.2

a) Starting with the commutation relation

$$[x_i, p_j] = i\delta_{ij}$$

and the definition of orbital angular momentum

$$\vec{L} = \vec{x} \times \vec{p},$$

derive the commutation relation for the latter, i.e.

$$[L_i, L_j] = i\epsilon(i, j, k)\, L_k.$$

b) Find the matrix representation of L_z, i.e. find the 3x3 matrix which brings about a rotation along the z-axis

$$D(\phi, [0, 0, 1]) = e^{-iL_z\,\phi} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The answer is

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[0 -im 0; im 0 0; 0 0 0]
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Verify this with a program. Also, find the eigenvalues of L_z.

c) Similarly, find L_x, L_y, and show that

$$[L_i, L_j] = i\epsilon(i, j, k)\, L_k.$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = 2\,I.$$

d) When two operators commute, we can find states which are simultaneously eigenstates of both operators, i.e., we can simultaneously measure them. Here we pick $|L^2; L_z\rangle$'s and label them according to the eigenvalues of the operators.

Verify the pattern of angular momentum:

$$|\ell(\ell + 1); m\rangle$$

with $m = -\ell, -\ell + 1, \ldots, \ell$.

Show that

$$L_\pm = L_x \pm i L_y$$

are the raising and lowering operators.

e) The spin operator also satisfies the commutation relation for angular momenta, i.e.

$$S = \frac{\vec{\sigma}}{2}$$
$$[S_i, S_j] = i\,\epsilon(i,j,k)\,S_k.$$

What do we get for the simultaneous eigenstates $|S^2; S_z\rangle$? Compare the results with $|L^2; L_z\rangle$ and explain the differences.

Q.3

Legendre Transform.

The Lagrangian $L(\dot{x})$ and the Hamiltonian $H(p)$ is related by a Legendre Transform:

$$H = p\dot{x} - L,$$

satisfying the canonical relations:

$$p = \frac{\partial L}{\partial \dot{x}}$$
$$\dot{x} = \frac{\partial H}{\partial p}.$$

a) Starting from the Hamiltonian of a free non-relativistic particle:

$$H = \frac{p^2}{2m},$$

derive an expression for the Lagrangian $L(\dot{x})$.

b) Do the same for a relativistic particle:

$$H = \sqrt{p^2 + m^2}.$$

Q.4

Gaussian integrals, generalized.

a) The Gaussian integral,

$$\int_{-\infty}^{\infty} dx\, e^{-A\,x^2} = \sqrt{\frac{\pi}{A}}$$

generalizes to

$$\int dx_1\, dx_2, \ldots dx_N\, e^{-\vec{x}^t A \vec{x}} = \sqrt{\frac{\pi^N}{\mathrm{detA}}}$$

.

Here A should be an $N \times N$, positive definite matrix, and

$$\vec{x}^t A \vec{x} = \sum_{i,j} x_i x_j A_{ij}$$

Write a program to verify the result for N = 2, 3, 6. Start with an A which is positive definite. (no diagonal please!)

b) Can you perform calculations for $N = 10$ with your method of choice?

Q.5

The Z's and the W's. (Method of auxiliary field)

a) Introducing an external field h to the Gaussian integral

$$Z(h) = \int_{-\infty}^{\infty} dx\, e^{-A\,x^2 + h\,x}.$$

Show that

$$Z(h) = \sqrt{\frac{\pi}{A}}\, e^{\frac{h^2}{4A}}.$$

b) If we define

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} dx\, x^n\, e^{-A\,x^2}}{\int_{-\infty}^{\infty} dx\, e^{-A\,x^2}},$$

show that

$$\langle x^n \rangle = Z_n = \frac{1}{Z(0)}\, \partial_h^n Z(h).$$

It is understood that we take $h \to 0$ at the end of the calculation. This allows us to obtain the various integrals by taking derivatives.

c) Compute (analytically and numerically): Z_n for $n = 1, 2, ..., 6$.

checks:

$$\begin{aligned} Z_2 &= \frac{1}{2A} \\ Z_4 &= \frac{3}{4A^2}. \end{aligned}$$

The general formula for Z_N $(N = 2k)$ is **obviously**

$$Z_{2k} = \frac{\mathrm{Gamma}(k + 1/2)}{\mathrm{Gamma}(1/2)} \times \frac{1}{A^k}.$$

Try to put $1/2, 3/4, 15/8, 105/16$ in here.

d) The connected piece.

Knowing all possible $\langle x^n \rangle$'s can tell us many things (if not everything) about the system. But deep down we know the system is simple, i.e. there is a lot of redundancy in Z_N's.

If we choose to describe the system with $W(h)$, defined as

$$W(h) = \ln Z(h).$$

And translate $W_M = \partial_h^M W(h)$ to and from Z_N, for example:

$$\begin{aligned} W_1 &= Z_1 \\ W_2 &= Z_2 - Z_1^2 \\ W_3 &= Z_3 - 3Z_1 Z_2 + 2Z_1^3 \\ W_4 &= Z_4 - 4Z_1 Z_3 - 3Z_2^2 + 12 Z_1^2 Z_2 - 6Z_1^4 \end{aligned}$$

(can you do more? and how to go from Z to W?)

In the language of W_N, the system is simple:

$$W(h) = \frac{h^2}{4A}.$$

and hence we have only W_1, W_2, and

$$W_{N>2} = 0.$$

in this Gaussian system.

Verify now

$$\begin{aligned} Z_2 &= W_2 = \frac{1}{2A} \\ Z_4 &= 3W_2^2 = \frac{3}{4A^2}. \end{aligned}$$

This exercise brings us dangerously close to field theory: In the language of field theory, $Z(h)$ is called a generator for Green's function, and $W(h)$ is called the generator for **connected** Green's function. In particular, W_2 is almost like a propagator.