HW4 due 05/01/2024

(+2 points for handing in on time)

Q.1

a) Prove the rotation formula of Pauli matrices:

$$D^{-1}\vec{\sigma}D = \cos\theta\,\vec{\sigma} + (1-\cos\theta)\,(\hat{n}\cdot\vec{\sigma})\;\hat{n} + \sin\theta\,\hat{n}\times\sigma.$$

where

$$D = e^{-i\frac{1}{2}\theta\,(\hat{n}\cdot\vec{\sigma})}$$

and $\hat{n}$ is a unit (3D) vector.

b) Pick $\hat{n} = [0,0,1]$, work out the analytic result and verify numerically.

Q.2

Story of a 2Der.

A smart physicist living in a 2D world has just discovered quantum mechanics. She writes down the operators she knows:

$$\vec{x} = [x, y]$$
$$p_x \to -i\partial_x$$
$$p_y \to -i\partial_y,$$

last, but not least, an orbital angular momentum operator

$$L = -i(x\partial_y - y\partial_x).$$

a) Verify the following commutation relations

$$[L, p_x] = ip_y$$
$$[L, p_y] = -ip_x.$$

b) Show that $T_x(h_x) = e^{-ih_x p_x}$ is the translation operator, i.e.,

$$T_x^{-1}\,x\,T_x = x + h_x,$$

and acting on a function in x-space like

$$\langle x|T_x(h_x)|f\rangle \to e^{-h_x\,\partial_x}\,f(x) = f(x-h_x).$$

c) Show that $D(\phi) = e^{-i\phi L}$ generates a rotation, i.e. (active rotation convention)

$$D^{-1}\vec{x}D = R\vec{x},$$
$$D^{-1}\vec{p}D = R\vec{p},$$

with

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

hint: use the relation

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots.$$

d) Verify the matrix representation for L reads

$$L \to i\frac{\partial}{\partial\phi}R = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(derivative evaluated at $\phi \to 0$)

e) While it is possible to write the generator of rotation as a matrix, it is NOT possible to do so for translation. Why is that?

f) You, being a student living in a 3D world, want to help the poor 2Der. You tell her that you can see beyond 2D and advise her to write R as R_3:

$$R_3 \to \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the translation operator T like this

$$T(h_x, h_y) \to \begin{bmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \\ 0 & 0 & 1 \end{bmatrix}.$$

But all her vector [x, y] should become [x, y, 1].

Demonstrate how the translation operator work. Find the generators and verify the commutation relations

$$[L, p_x] = ip_y$$
$$[L, p_y] = -ip_x,$$

now as matrix relations, are satisfied.

g) Verify (with the 3D matrix representation) the relations:

$$R_3^{-1} p_x R_3 = \cos(\phi)\, p_x - \sin(\phi)\, p_y$$
$$R_3^{-1} p_y R_3 = \cos(\phi)\, p_y + \sin(\phi)\, p_x.$$

In fact we 3Ders have the same problem as the 2Der: translation in 4D is NOT a linear transformation. This takes us from Lorentz group to Poincare group, and we need 5 dimensional matrix to handle that.

Q.3 Numerical solution to the Schroedinger Equation.

Solve the Schroedinger Equation numerically and compute the energy spectrum (and the wavefunctions) of the Morse Potential:

$$V(x) = \frac{\lambda^2}{2m}\,(e^{-2x} - 2 * e^{-x}).$$

- a) (choose e.g. $m = 1$, $\lambda = 6$) Compute the ground state and the first 4 excited states. Check with the analytic result for the energy levels:

$$E(n) = -(\lambda - 1/2 - n)^2\,\frac{1}{2m}.$$

- b) Approximate the Morse potential with a harmonic potential. Check that their ground states agree. What about the 1st excited state?

Q.4 Numerical study of 3D Ising Model

- a) Write a program to solve the 3D Ising Model. What is the critical temperature you get?
- b) Plot the average spin and the spin susceptibility as functions of temperature.