HW5 due 26/01/2024

(+2 points for handing in on time)

Q.1

Consider the function of iterated exponential

$$h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} .$$

a) The function $h(x)$ admits a recursive definition via

$$h(x) = x^{h(x)} \quad .$$

Write a program to implement such a construction.

b) Another way is construct $h(x)$ is by inverting the relation

$$x(h) = e^{\frac{1}{h}\,\ln h} \quad .$$

Plot h versus x using this and the recursive construction. Compare the results. Note in particular the endpoint at $(x = e^{1/e}, h = e)$.

c) What is the result of i^i? What about i^{i^i}? Check with numerical construction.

d) Iteratively construct $h(i)$ and plot the sequence of complex numbers in the complex plane. What a beautiful pattern! Merry Xmas and Happy New Year!

e) The Lambert function $W(x)$ is defined by the relation

$$W(x)\, e^{W(x)} = x.$$

Study (and plot) the function $x(W)$, show that the minimum is located at

$$W = -1$$
$$x = -\frac{1}{e}.$$

Show that

$$h(x) = e^{-W(-\ln x)}.$$

Explain the endpoint of $h(x)$ identified.

Q.2 Matsubara sum.

a) Study (again!) the basic integral

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \frac{1}{p_4^2 + \omega^2} = \frac{1}{2\omega}.$$

Consider

$$p_4 = n\Delta p_4$$
$$\Delta p_4 = \frac{2\pi}{L_4},$$

where $n = -N_{\text{max}}, -N_{\text{max}}+1, \ldots, N_{\text{max}}$ and convert the integral into a Riemann sum

$$\frac{1}{L_4} \sum_{n=-\infty}^{\infty} \frac{1}{(\Delta p_4)^2 \; n^2 + \omega^2}.$$

Numerically evaluate the sum (for large enough N_{max} and small enough Δp_4) and verify the integral.

b) The case of a finite L_4 is an important result in finite temperature field theory: $L_4 \to \beta = 1/T$ is the inverse temperature, $p_4 \to \omega_n = \frac{2\pi}{\beta} n$ are the Matsubara frequencies (for Bosons), and the sum of interest reads

$$G(\beta, \omega) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \omega^2}.$$

The analytic result is

$$G(\beta, \omega) = \frac{1}{2\omega} \coth(\frac{\beta\omega}{2}).$$

To derive this result, we can use the Euler's product formula for $\sin(x)$

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right).$$

Make sense of the sine formula by numerically computing the RHS (for a large N_{max}) and plotting the two functions.

c) Show that

$$\sin(ix) = i\sinh(x) = i\frac{e^x - e^{-x}}{2}$$

and obtain an analogous product formula for $\sinh(x)$. The result is

$$\sinh(x) = x\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2\pi^2}\right).$$

d) Take the logarithm on both sides and reach

$$\ln\sinh(x) = \ln x + \sum_{n=1}^{\infty}\ln\left(1 + \frac{x^2}{n^2\pi^2}\right).$$

Now take the derivative (with respect to x) on both sides and show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2\pi^2 + x^2} = \frac{1}{x}\coth(x).$$

e) Finish the job to show that

$$G(\beta,\omega) = \frac{1}{2\omega}\coth(\frac{\beta\omega}{2}).$$

What is the $\beta \to \infty$ limit? Explain.

Q.3

a) From $S = e^{2i\delta} = 1 + it$, show that $2\text{Im}t = |t|^2$.
b) Given the scattering amplitude for a resonant process can be parametrized as

$$f(E) \propto \frac{1}{E - E_R + i\frac{\gamma}{2}},$$

show that the resonant phase shift $\delta_R(E)$ satisfies

$$\tan\delta_R(E) = \frac{-\gamma/2}{E - E_R}.$$

Plot the phase shift function with some reasonable values of the parameters.

Q.4

Derive the key results in the scattering theory:

a) The Green's function: show that

$$
\begin{aligned}
G_E^0(x) &= \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \frac{1}{E - \frac{q^2}{2m_R} + i\delta} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty dq\, \sin(qr)\, \frac{q}{E - \frac{q^2}{2m_R} + i\delta} \\
&= -2m_R \times \frac{1}{4\pi r}\, e^{+ipr}.
\end{aligned}
$$

b) Show that the G_E^0 satisfies

$$(E - \hat{H}_0) G_E^0(\vec{x_1}, \vec{x_2}) = \delta^{(3)}(\vec{x_1} - \vec{x_2})$$

where $\hat{H}_0 = -\frac{\nabla^2}{2m_R}$.

c) The full Green's function G_E satisfies

$$(E - \hat{H}) G_E(\vec{x_1}, \vec{x_2}) = \delta^{(3)}(\vec{x_1} - \vec{x_2}).$$

where $\hat{H} = \hat{H}_0 + \hat{V}$. Derive the relation between G_E and G_E^0.

d) What about the full wavefunction ψ and the scattering amplitude f? How can they be extracted from G^0?

Q.5

a) Numerically compute the S-wave phase shift $\delta(E)$ for a finite barrier:

$$V(r) = V_0,$$

if $x < R$, otherwise zero. You can take $m_R = 1$, $V_0 = 2.0$, $R = 1.5$. Plot the result in a suitable energy range: e.g. 0.1:14.0.

b) The effective spectral function

$$\Delta A(E) = 2\, \frac{d}{dE} \delta(E)$$

corresponds to the change of the density of states due to interaction. Compute $\Delta A(E)$ for the finite barrier problem.